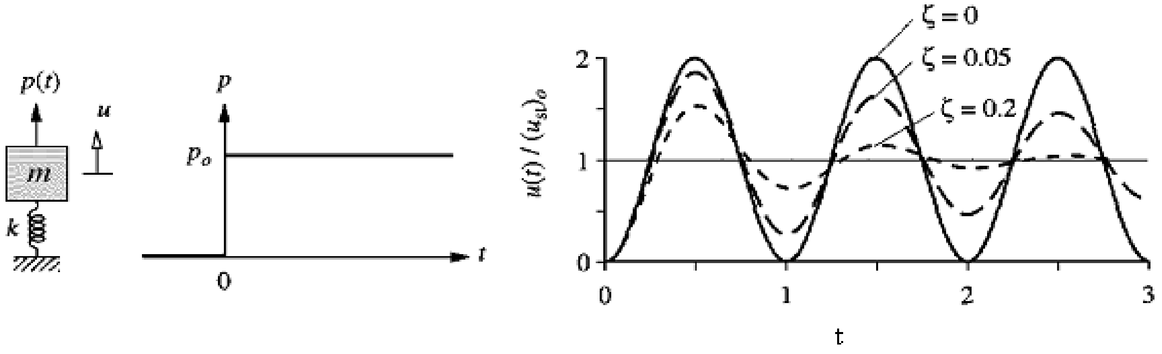
Theory:

Here 6 types of Impulse forces are considered. They are,

1. Step force:

A step force jumps suddenly from zero to p_0 and stays constant at value. It is desired to determine the response of an undamped SDF system. Starting at rest to step force:

$p(t) = p_0$



Where $(u_{st})_0 = \frac{p_0}{k}$, the static deformation due to force p_0

Equation of motion for this step force:

$$u(t) = e^{-\zeta\omega_n t}(A\cos\omega_D t + B\sin\omega_D t) + \frac{p_0}{k}$$

where

$$A = -\frac{p_0}{k}\ ,\ B = -\frac{p_0}{k}\ \frac{\zeta}{\sqrt{1-\zeta^2}}$$

$\omega_D = damped\ frequency$

$\zeta = damping\ ratio$

k = stiffness

$\omega_n = natural\ frequency$

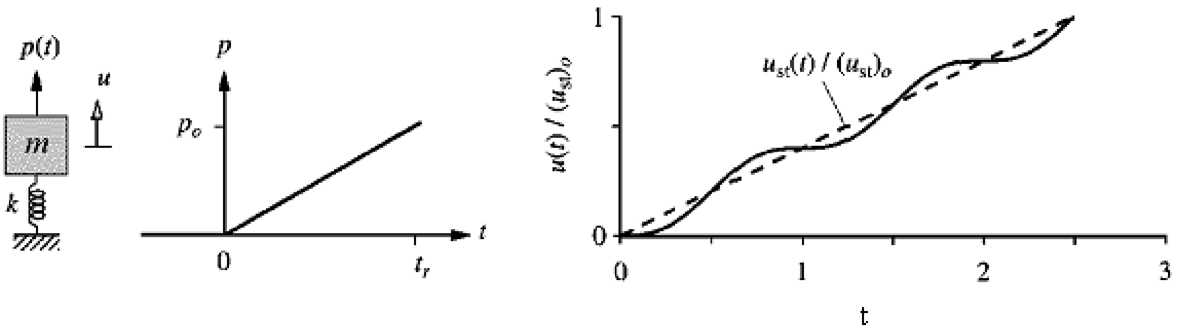
2. Ramp force:

The applied force p(t) increases linearly with time. Naturally, it cannot increase indefinitely, but our interest is confined to the duration where p(t) is still small enough that resulting spring force is within the linearly elastic limit of the spring.

$$P(t) = p_0 \frac{t}{t_r}$$

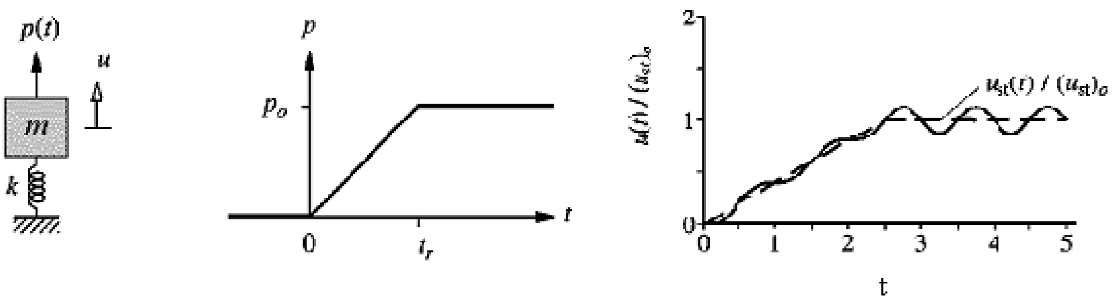
$$u(t) = \frac{1}{m\omega_n} \int_0^1 \frac{p_0}{t_r} \tau \sin \omega_n (t - \tau) d\tau$$

$$u(t) = (u_{st})_0 \left(\frac{t}{t_r} - \frac{\sin \omega_n t}{\omega_n t_r} \right)$$



3. Step force with finite rise time:

Since in reality a force can never be applied suddenly. It is of interest to consider a dynamic force that has a finite rise time, t_r, but remains constant thereafter.



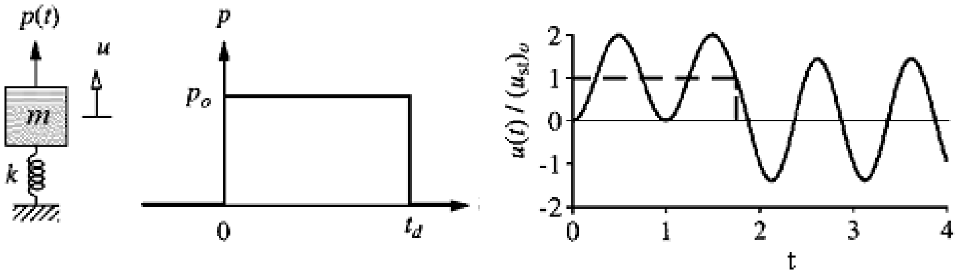
$$P(t) = \begin{cases} p_0 \left[\frac{t}{t_r}\right] & t \le t_r \\ p_0 & t \ge t_r \end{cases}$$

$$u(t) = u(t_r) \cos \omega_n (t - t_r) + \frac{\dot{u}(t_r)}{\omega_n} \sin\omega_n (t - t_r) + (u_{st})_0 [1 - \cos\omega_n (t - t_r)]$$

$$u(t) = (u_{st})_0 \left\{ 1 - \frac{1}{\omega_n t_r} [\sin\omega_n t - \sin\omega_n (t - t_r)] \right\}$$

4. Rectangular pulse force:

$$m\ddot{u} + ku = p(t) = \begin{cases} p_0 & t \le t_d \\ 0 & t \ge t_d \end{cases}$$



With at-rest initial conditions: $u(0) = \dot{u}(0) = 0$.The analysis is organized in two phases.

1. **Forced vibration phase**: During this phase, the system is subjected to a step force. The response of the system is given

   $u(t) = (u_{st})_0$

   $$\frac{u(t)}{(u_{st})_0} = 1 - \cos\omega_n t = 1 - \cos\frac{2\pi t}{T_n}; \qquad t \le t_d$$

2. **Free vibration phase**: After the force ends at t_d ,the system undergoes free vibration, defined by modifying appropriately:

$$u(t) = u\,(t_d)\ \cos\omega_n\,(t - t_d) + \frac{\dot{u}(t_d)}{\omega_n}\sin\omega_n\,(t - t_d)$$

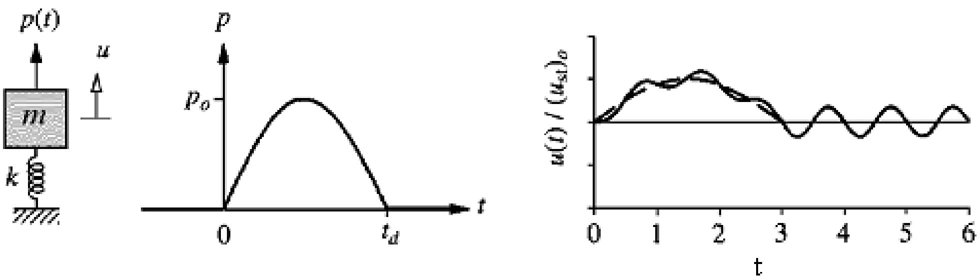
$$\frac{u(t)}{(u_{st})_0} = (1-\cos\omega_n\, t_d)\ \cos\omega_n\,(t - t_d) + \sin\omega_n\, t_d\ \sin\omega_n\,(t - t_d)\ ; \quad t \geq t_d$$

$$\frac{u(t)}{(u_{st})_0} = \cos\omega_n\,(t - t_d) - \cos\omega_n\, t\ ; \qquad t \geq t_d$$

5. Half –cycle sine pulse force:

The next pulse we consider is a half-cycle of sinusoidal force. The response analysis procedure for this pulse is the same as developed for rectangle pulse, but the mathematical details become more complicated.

$$m\ddot{u} + ku = p(t) = \begin{cases} p_0\ \mathrm{Sin}\left(\frac{\Pi t}{t_d}\right) & t \leq t_d \\ 0 & t \geq t_d \end{cases}$$



Case 1: $\frac{t_d}{T_n} \neq \frac{1}{2}$

Forced vibration phase: The force is the same as the harmonic force p(t) = p_0 sin ωt considered earlier with frequency $\omega = \frac{t_d}{T_n}$.

$$\frac{u(t)}{(u_{st})_0} = \frac{1}{1-\left(\frac{T_n}{2t_d}\right)^2}\left[\sin\left(\Pi\frac{t}{t_d}\right) - \frac{T_n}{2t_d}\sin\left(2\Pi\frac{t}{t_d}\right)\right] \qquad t \leq t_d$$

Free vibration phase: After the force pulse ends, the system vibrates freely with its motion.

$$\frac{u(t)}{(u_{st})_0} = \frac{\frac{T_n}{t_d}\cos\frac{\Pi t_d}{T_n}}{\left(\frac{T_n}{2t_d}\right)^2 - 1} \text{ Sin}\left[2\Pi\left(\frac{t}{t_n} - \frac{t_d}{2T_n}\right)\right] \quad ; \quad t \geq t_d$$

Case 2 : $\frac{t_d}{T_n} = \frac{1}{2}$

Forced vibration phase: The force is now given by equation

$$\frac{u(t)}{(u_{st})_0} = \frac{1}{2}\left(sin\frac{2\Pi t}{T_n} - \frac{2\Pi t}{T_n}\cos\frac{2\Pi t}{T_n}\right) \quad ; \quad t \leq t_d$$

Free vibration Phase: After the force pulse ends at $t = t_d$, free vibration of the system is initiated by the displacement u(t_d) and velocity $\dot{u}(t_d)$ at the end of the force pulse.

$$\frac{u(t)}{(u_{st})_0} = \frac{\Pi}{2}\cos 2\Pi\left(\frac{t}{T_n} - \frac{1}{2}\right) \quad ; \quad t \geq t_d$$